Equilibrium Configurations and Energies of the Rotating Elastic Cable in Space

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A freely rotating elastic rod or cable in space may have a variety of equilibrium shapes, both intersecting and nonintersecting. The nonlinear governing equations are integrated numerically to determine the intersecting configurations. Minimum energy and rotation rate requirements for their existence are found.

Introduction

FLEXIBLE rods or cables in space such as tethers and antennas are becoming longer due to necessity. For example, the Radio Astronomy Explorer Satellite[1] used 750-ft. antennas for detecting low-frequency signals. On the other hand, the rotation of space structures are sometimes desirable and at other times unavoidable. In a recent paper[2] the equilibrium configurations of a freely rotating thin elastic rod were studied, perhaps for the first time. However, only nonintersecting configurations were considered. The purpose of this paper is to investigate whether there are other forms of equilibrium configurations and what would be the requirements for their existence. These will be determined through a study of the energy contained.

Formulation

Consider an originally straight, thin elastic rod of length ℓ and uniform density ρ. Figure 1 shows a possible configuration where the rod is in steady rotation with angular velocity Ω. Let axes x', y' be in the plane of the rotating rod with $x'=0$ at one end and the x' axis along the axis of rotation. For thin rods the local moment m is proportional to the local curvature

$$m = -EI\frac{\mathrm{d}\theta}{\mathrm{d}s} \tag{1}$$

where EI is the flexural rigidity, θ is the local angle of inclination and s' is the arc length from one end. A local balance of moment gives

$$m = m + \mathrm{d}m + \left(\int_0^{s'} \rho\Omega^2 y'\,\mathrm{d}s'\right)\mathrm{d}s'\cos\theta \tag{2}$$

The governing equations become

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = -J^4 u\cos\theta \tag{3}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}s^2} = -\sin\theta \tag{4}$$

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = -\sin\theta \tag{5}$$

where

$$u \equiv \int_0^s y\,\mathrm{d}s \tag{6}$$

Here the unprimed lengths have been normalized with respect to ℓ and

$$J \equiv \ell\rho^{1/4}\Omega^{1/2}(EI)^{1/4} \tag{7}$$

represents an important nondimensional parameter signifying the relative importance of length, density, and rotation to flexural rigidity. The boundary conditions are that the moments and forces are zero at the ends.

The Equilibrium Configurations

The nonintersecting configurations have been determined previously[2] and will only be discussed briefly here. In general, these cases bifurcate from the axially rotating straight rod when the rotation rate is larger than certain critical speeds. Higher values of J lead to the more sinuous configurations, Figure 2 shows that the equilibrium shapes may be either symmetric or antisymmetric with respect to the midperpendicular plane. Static stability analysis shows these configurations are the only ones which bifurcate from the straight rotating rod.

Is it possible for the rotating rod to exist in other equilibrium shapes? The answer is yes, if the shape of the rod intersects itself. Figure 3 shows three such possibilities, all locally stable to small disturbances. These intersecting cases do not bifurcate from the straight rotating rod and cannot be predicted from linear stability analysis.

For case III in Fig. 3 where the configuration intersects itself once and is symmetrical about the axis of rotation, we get

$$t \equiv SJ \qquad v(t) \equiv -J^2 u(s) \tag{8}$$

Equations (3) and (7) become

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = v\cos\theta \qquad \frac{\mathrm{d}^2 v}{\mathrm{d}t^2} = \sin\theta \tag{9}$$

The initial conditions are

$$v(0) = 0 \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t}(0) = 0 \tag{10}$$

For given $\theta(0)$ we guess $\mathrm{d}v/\mathrm{d}t(0)$ and integrate Eq. (9) until θ becomes -90 deg, say at $t = t^*$. A solution is found if $\mathrm{d}v/\mathrm{d}t$ is zero there. If not, $\mathrm{d}v/\mathrm{d}t(0)$ is adjusted. By one-parameter shooting, we find

$$J = 2t^* \tag{11}$$

Received Oct. 14, 1985; revision received March 15, 1986.

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$$y(0)=\frac{du}{ds}(0)=-\frac{1}{2t^*}\,\frac{dv}{dt}(0) \tag{12}$$

The equilibrium configuration can be obtained from integrating Eqs. (3-5) once with the initial conditions

$$u(0)=\frac{d\theta}{ds}(0)=x(0)=0 \tag{13}$$

and the values of $\theta(0)$, J, $y(0)$, $du/ds(0)$ are obtained from Eqs. (11) and (12).

The configuration of case IV is symmetrical about a line normal to the rotation axis. The integration method is similar. The only difference is the end condition. When θ decreases to -180 deg, we check to see whether v is zero.

Case V is more complicated since there is no symmetry. The best way is to integrate until v becomes zero the third time and then check whether $d\theta/dt$ is zero there, say at $t=\tilde{t}$. If so

$$J=\tilde{t} \tag{14}$$

$$y(0)=-\frac{1}{\tilde{t}}\,\frac{dv}{dt}(0) \tag{15}$$

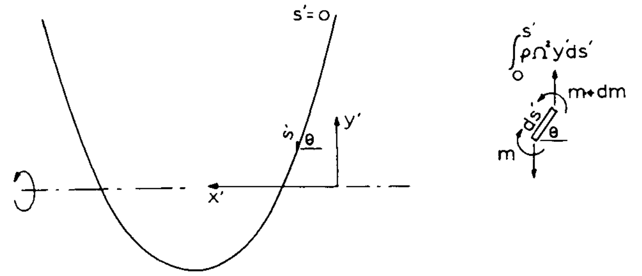

Fig. 1 The coordinate system.

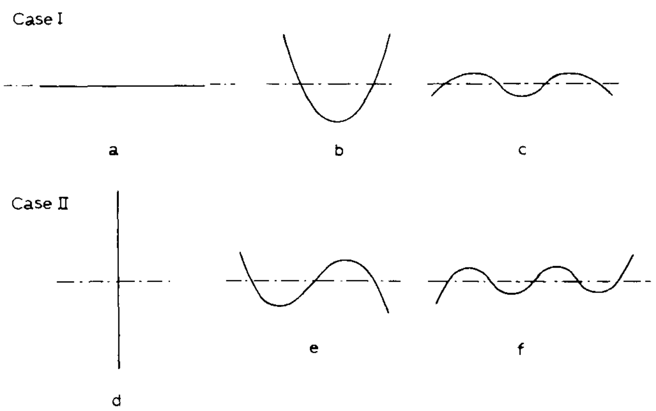

Fig. 2 Case I and case II configurations.

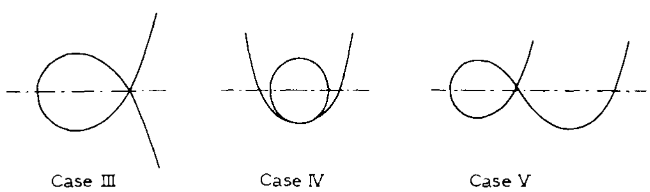

Fig. 3 Case III, IV, V configurations (each intersects itself once).

Figure 4 shows the results for $y(0)$ (maximum amplitude) as a function of the rotation parameter J. Note that the curves do not bifurcate from the $y=0$ axis, thus cannot be predicted from linear stability theory. Let us look at case III solutions. For low rotation speeds ($0\leq J<6.92$), equilibrium cannot exist. At $J=6.92$, a state (point A) with large deformation [$y(0)=0.27$] is possible.A further increase in J yields two different equilibrium solutions. The amplitude of the states represented by the branch AB which increases with rotation and is asymptotic to the 0.5 line. The shape gradually becomes a straight line except for a loop near the rotation axis. The branch ACD decreases with increased J and is asymptotic to 0.166 which is half the length of a rod folded into thirds. When J is larger than 10.2, case IV is possible; and when J is larger than 10.83, case V configurations may exist. Both case IV and case V have two branches and their amplitudes are asymptotic to 0.25 and 0.125.

Figure 5 shows some configurations for case III. The letters correspond to the states represented in Fig. 4. State A requires the lowest rotation rate J for all intersecting configurations. States B and C have the same rotation J but are on different solution branches. State D is of higher J on the same branch as state C. Figure 6 shows the equilibrium configurations for case IV solutions. A higher rotation rate is required for case IV. The shapes of the elastic cable for case V are shown in Fig. 7.

Figure 8 shows the maximum normalized force as a function of J represented by u occurring at the first $y=0$ point. Figure 9 depicts the normalized maximum moment which is also the

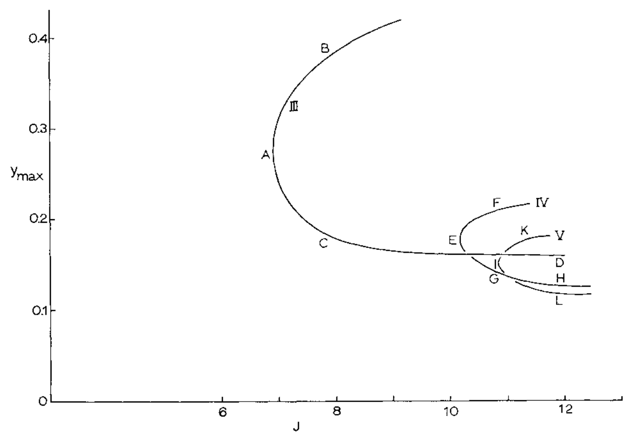

Fig. 4 The maximum amplitude as a function of J.

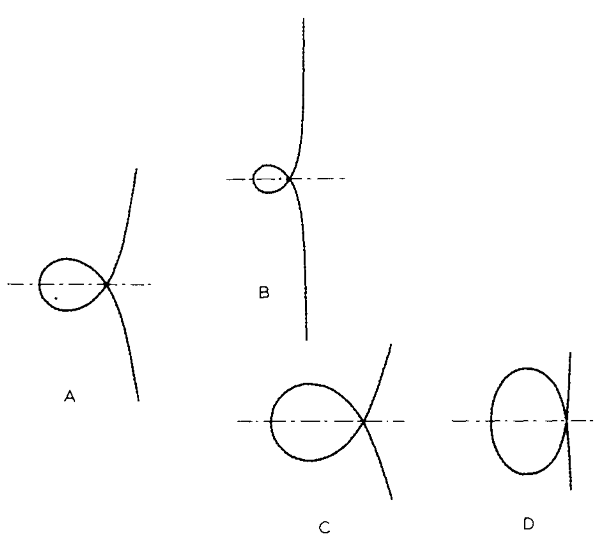

Fig. 5 Configurations of case III, A: $J=6.92$; B: $J=7.88$; C: $J=7.88$; D: $J=12.03$. (Letters correspond to states in Fig. 4.)

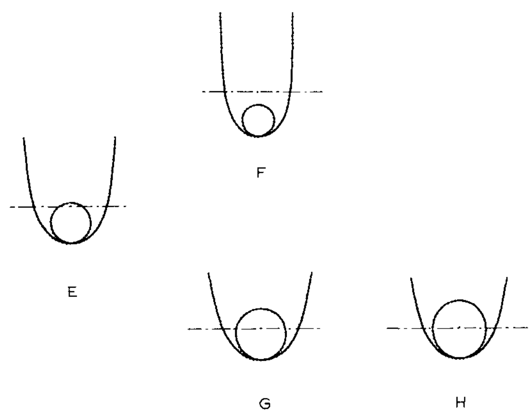

Fig. 6 Configurations of case IV, E: $J=10.20$; F: $J=10.76$; G: $J=10.76$; H: $J=11.93$.

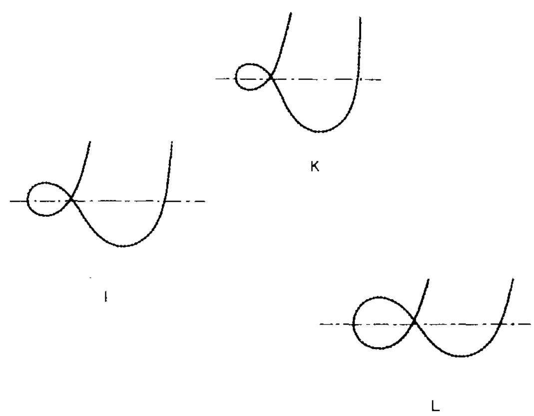

Fig. 7 Configurations of case V, I: $J=10.85$; K: $J=11.28$; L: $J=11.88$.

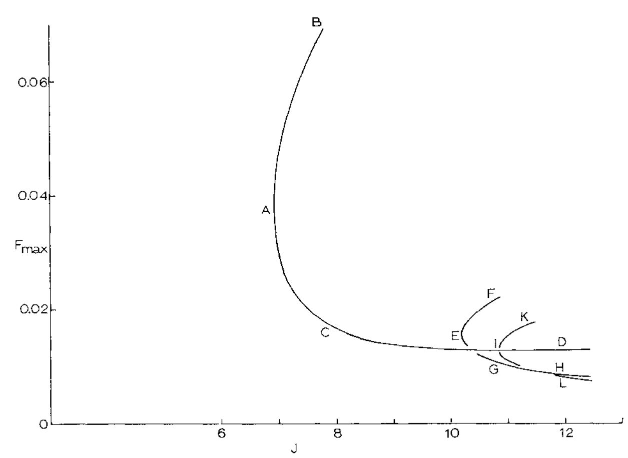

Fig. 8 The maximum force.

maximum local curvature. For case III configurations, the maximum moment occurs at the midpoint ($s=0.5$) for the states along CAB. For the segment CD the maximum moment occurs at a point near $\theta=0$ and $-\pi$. For case IV the maximum moment occurs at the midpoint. For case V the maximum moment is near $\theta=\pi/2$. Figures 8 and 9 are useful in the design of rotating flexible cables.

Energy

The total energy contained in a rotating rod is the sum of the kinetic energy due to rotation and the strain energy due to

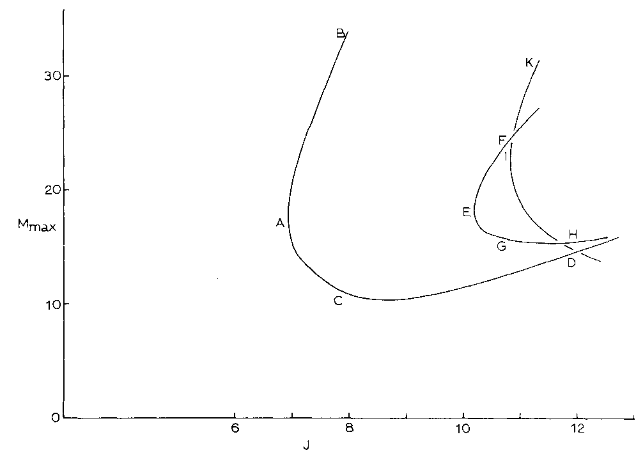

Fig. 9 The maximum local moment.

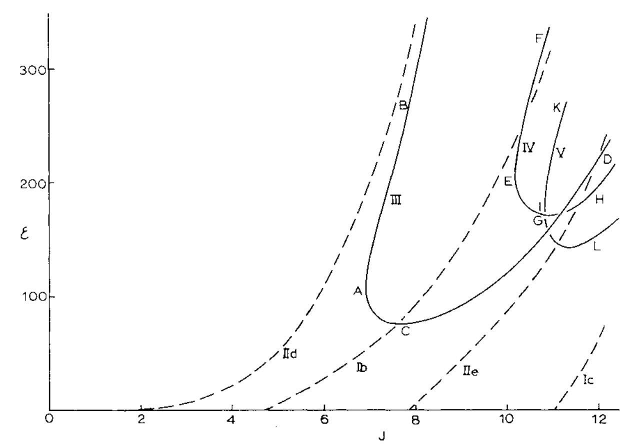

Fig. 10 The total energy.

deformation. The total energy, normalized by $EI/(2\ell)$, is

$$\epsilon=\int_0^1 J^4 y^2(s)\,\mathrm{d}s+\int_0^1\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2\mathrm{d}s \tag{16}$$

Since we already obtained $\theta(s)$ and $y(s)=\mathrm{d}u/\mathrm{d}s$, Eq. (16) can be integrated. An easier way is to write Eq. (16) as a differential equation

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}s}=J^4\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2+\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 \qquad \epsilon(0)=0 \tag{17}$$

and solve numerically with Eqs. (3) and (4). The results are shown in Fig. 10 together with the total energy of the nonintersecting cases (Fig. 2) shown as dashed lines. The axially rotating straight rod (case Ia) has zero total energy, since the rod is assumed to be infinitesimally thin. The straight rod rotating about a line normal to its axis (case IId) has kinetic energy $J^4/12$ and zero strain energy. The symmetric hairpin shape (case Ib) bifurcates from $J=4.7300$ and the antisymmetric "S" shape (case IIe) bifurcates from $J=7.8532$. These bifurcation values are twice the values from Ref. 2 since that source defines J with half the length of the rod. We see that the energy of nonintersecting shapes bifurcate starting from $\epsilon=0$, while the intersecting cases (III, IV, V) require a finite amount of energy levels before they can be realized.

Let us study the curve for case III, which is the most important intersecting configuration. The foremost requirement for case III to exist is that a minimum total energy of $\epsilon=76$ (occuring at state C) must be imported to the rotating rod. On the other hand, the rotation rate should be greater than $J=6.92$ (occuring at state A). The branch AB becomes asymptotic to case Id since their shapes become similar as J is increased. The branch CD eventually will merge with the curve for case IIe. Both branches of AB and CD have positive slope and are in stable equilibrium. The segment AC has negative slope (an increase in rotation rate required a decrease in total energy) and is therefore unstable. Similarly case IV and case V also have segments of negative slope which are unstable. For case IV, we require $\epsilon > 172$, and for case V, $\epsilon > 142$. We expect rods with two or more self-intersections (not studied here) would require such a high total energy that they would rarely occur in practice.

References

[1]Stone, R. G., "RAE-1500 ft Antenna Satellite," *Astronautics and Aeronautics*, Vol. 3, March 1965, pp. 46–49.

[2]Wang, C. Y., "Rotation of a Free Elastic Rod," *Journal of Applied Mechanics*, Vol. 49, March 1982, pp. 225–227.